On optimal weighted-sum rates for the modulo sum problem


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Distributed Source Coding

Let (X^n, Y^n) be a sequence of random variables that are generated i.i.d. according to $p(x, y)$, and denote $Z^n = (f(X_1, Y_1), f(X_2, Y_2), \cdots, f(X_n, Y_n))$.

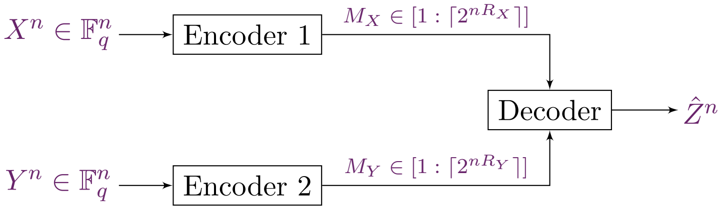

Figure 1: Distributed Source Coding

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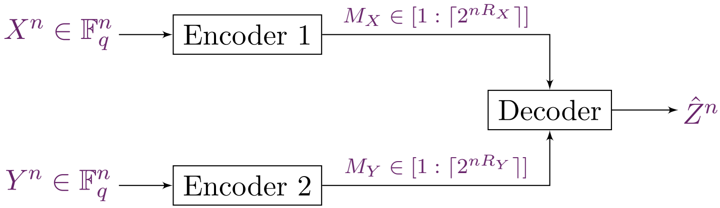

Figure 1: Distributed Source Coding

A rate pair (R_X, R_Y) is said to be achievable if there exists a sequence of (n, R_X, R_Y)-codes such that the $Pr(\hat{Z}^n \neq Z^n) \to 0$ as $n \to \infty$.

Problem

Consider $f(X_i, Y_i) = X_i \oplus Y_i$, what rate pairs (R_X, R_Y) are achievable?

Known results

Slepian-Wolf region: [Slepian-Wolf 73']

When $Z = (X, Y)$, the capacity region is given by

$$\begin{aligned} R_X &\geq H(X|Y) \\ R_Y &\geq H(Y|X) \\ R_X + R_Y &\geq H(XY) \end{aligned}$$

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Recovering (X, Y) with high probability is sufficient to decode any function $f(X, Y)$.

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Körner-Marton region: [Körner-Marton 79']

When $Z = X \oplus Y$, a rate pair (R_X, R_Y) is achievable by linear codes if

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This matches capacity region when (X, Y) follows Doubly Symmetric Binary Source distribution.

Remark:

First example showing that structured codes outperforms random coding for multiuser information theory problems.

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More distributions are discovered for optimality of this region on weighted sum rate. (this work)

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First example showing that structured codes outperforms random coding for multiuser information theory problems.

Known results

Exercise 16.23 in [Csiszár-Körner][1]

When $H(Z) \geq \min\{H(X), H(Y)\}$, Slepian-Wolf rate region characterizes the capacity region $\mathcal{C}$ for $Z = X \oplus Y$ in $GF(2)$.

[1] I. Csiszár and J. Körner, Information theory: Coding theorems for discrete memoryless systems

Known results

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More distributions are discovered for optimality of Slepian-Wolf region on weighted sum rate. (this work)

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More distributions are discovered for optimality of Slepian-Wolf region on weighted sum rate. (this work)

Ahlswede-Han region: [Ahlswede-Han 83']

When $Z = X \oplus Y$, a rate pair (R_X, R_Y) is achievable via a combination of linear codes and random binning if

$$R_X \geq I(U;X|V) + H(Z|UV)$$
$$R_Y \geq I(V;Y|U) + H(Z|UV)$$
$$R_X + R_Y \geq I(UV;XY) + 2H(Z|UV)$$

for some U and V that satisfy the Markov chain $U \to X \to Y \to V$.

[1] I. Csiszár and J. Körner, Information theory: Coding theorems for discrete memoryless systems

Known results

Cut-set lower bound: [Körner-Marton 79']

Any achievable rate pair (R_X, R_Y) for the modulo sum problem must satisfy

$$R_X \geq H(Z|Y) = H(X|Y)$$
$$R_Y \geq H(Z|X) = H(Y|X)$$
$$R_X + R_Y \geq H(Z).$$

Main result: A lower bound

Theorem 1

Any achievable rate pair (R_X, R_Y) for the modulo sum problem must satisfy the following constraints for any $\lambda \geq 1$:

$$R_X + \lambda R_Y \geq H(XY) + \min_{U \to X \to Y} \lambda H(Z|U) - H(Y|U)$$
$$\lambda R_X + R_Y \geq H(XY) + \min_{V \to Y \to X} \lambda H(Z|V) - H(X|V)$$

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Remark: From [Nair 10' 13']

$$\min_{U \to X \to Y} \lambda H(Z|U) - H(Y|U) = -\mathfrak{C}_{\mu(x)}[H(Y) - \lambda H(Z)]\Big|_{p(x)},$$

where $\mathfrak{C}_x[f]\big|_{x_0}$ denotes the upper concave envelope of the function $f(x)$ with respect to x evaluated at $x = x_0$.

Proof sketch

The multi-letter extensions of Ahlswede-Han region tends to the capacity region.

$$\begin{aligned} & n(R_X + \lambda R_Y) + n\varepsilon \\ & \geq I(U; X^n) + \lambda I(V; Y^n|U) + (1+\lambda) H(Z^n|UV) \end{aligned}$$

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Sinlge-letterize the lower bound

Lemma 1

Let $\lambda \geq 1$ and let (X^n, Y^n) be i.i.d distributed according to $p(x, y)$ where X, Y take values in a finite field. Let Z^n be obtained as $Z_i = X_i \oplus Y_i, i = 1, .., n$, i.e. the component-wise modulo sum on the field. Then for any $\lambda \geq 1$ the following holds:

$$\min_{\hat{U}: \hat{U} \to X^n \to Y^n} \lambda H(Z^n|\hat{U}) - H(Y^n|\hat{U}) = n \left(\min_{U: U \to X \to Y} \lambda H(Z|U) - H(Y|U) \right).$$

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Proof sketch

- Taking i.i.d. copies of the minimizer of the right hand side, left hand side is at most the value of right hand side.
- The other direction follows from Markov chain $(\hat{U}, Y_{i+1}^n, Z^{i-1}) \to X_i \to (Y_i, Z_i)$ and Körner-Marton identity.

Conditions for lower bound to be tight

Lemma 2

The lower bound for the weighted sum-rate $R_X + \lambda R_Y$, for $\lambda \geq 1$ given in Theorem 1 is optimal, i.e. matches the weighted sum-rate of the capacity region, if either of the following conditions hold:

(i) $\mathfrak{C}_{\mu(x)}[H(Y) - \lambda H(Z)]\big|_{p(x)} = H(Y) - \lambda H(Z)$ and Y is independent of Z,

(ii) $\mathfrak{C}_{\mu(x)}[H(Y) - \lambda H(Z)]\big|_{p(x)} = H(Y|X) - \lambda H(Z|X)$.

Further if condition (i) holds for some $\lambda_1 > 1$, then it will also hold for $1 \leq \lambda \leq \lambda_1$; and if condition (ii) holds for some $\lambda_2 \geq 1$, then it will also hold for $\lambda \geq \lambda_2$.

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Remark:

A relatively easier condition to verify is the convexity of $H(Y) - \lambda H(Z)$ with respect to the distribution of X.

Application to binary alphabets GF(2)

Notation: We will parameterize the space of distributions over pairs of binary alphabets, $p(x, y)$ as follows:
$P(X = 0) = x, P(Y = 0|X = 0) = c, P(Y = 1|X = 1) = d.$

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Proposition 1: Optimality of Slepian-Wolf region

The optimal weighted sum-rate of the capacity region is given by the Slepian Wolf region if any of the following conditions hold:

(*i*) For any λ, if $(c - \frac{1}{2})(d - \frac{1}{2}) \leq 0$, or

(*ii*) $\lambda \geq \left(\frac{c-\bar{d}}{c-d}\right)^2$, $c \neq d$, and $(c - \frac{1}{2})(d - \frac{1}{2}) > 0$.

where $\bar{d} = 1 - d$.

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Remarks:

(i) The condition (i) above is already known and stated as exercise 16.23 page 390 of Csiszár and Körner's book. One can verify that that $H(Z) \geq H(Y)$ is equivalent to $(c - \frac{1}{2})(d - \frac{1}{2}) \leq 0$.

(ii) Note that an equivalent proposition can also be stated for the alternate parameterization: $P(Y = 0) = y, P(X = 0|Y = 0) = \hat{c}, P(X = 1|Y = 1) = \hat{d}$.

Application to binary alphabets GF(2)

Proposition 2: Optimality of linear codes

Let $P(X=0)=x, P(Y=0|X=0)=c, P(Y=1|X=1)=d$ where $x=\frac{\sqrt{d\bar{d}}}{\sqrt{d\bar{d}}+\sqrt{c\bar{c}}}$.
The optimal weighted sum-rate of the capacity region is given by the Körner-Marton region, i.e. using linear codes, if any of the following conditions hold:

- (i) For any λ, if $c=d$, or
- (ii) $1 \le \lambda \le \lambda_1$, $c \ne d$, and $(c-\frac{1}{2})(d-\frac{1}{2})>0$, where λ_1 is the larger root of the quadratic equation

$$\lambda^2(c-d)^2+\lambda(2(c-d)(c-\bar{d})-4d\bar{d}(c-\bar{c})^2)+(c-\bar{d})^2=0.$$

where $\bar{d}=1-d, \bar{c}=1-c$.

Application to binary alphabets GF(2)

Proposition 2: Optimality of linear codes

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The optimal weighted sum-rate of the capacity region is given by the Körner-Marton region, i.e. using linear codes, if any of the following conditions hold:

- (i) For any λ, if $c=d$, or
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where $\bar{d}=1-d, \bar{c}=1-c$.

Remarks:

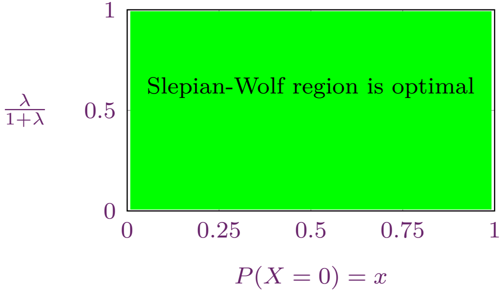
- (i) As long as $(c-\frac{1}{2})(d-\frac{1}{2})>0$ and $x=\frac{\sqrt{d\bar{d}}}{\sqrt{d\bar{d}}+\sqrt{c\bar{c}}}$, the optimal sum-rate will be given by the Körner-Marton region, i.e. using linear codes.
- (ii) An equivalent Proposition can also be stated for the alternate parameterization: $P(Y=0)=y, P(X=0|Y=0)=\hat{c}, P(X=1|Y=1)=\hat{d}$.

Application to binary alphabets GF(2)

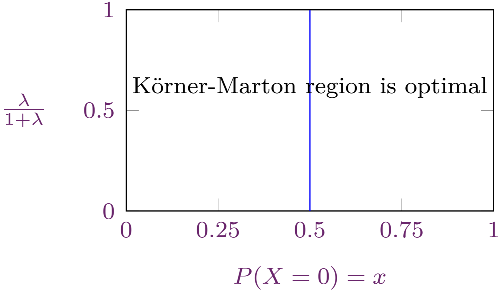
Notation: $P(X=0)=x, P(Y=0|X=0)=c, P(Y=1|X=1)=d$.

Known results: Optimality of weighted sum rates.

When $(c-\frac{1}{2})(d-\frac{1}{2}) \le 0$,



When $c=d$,

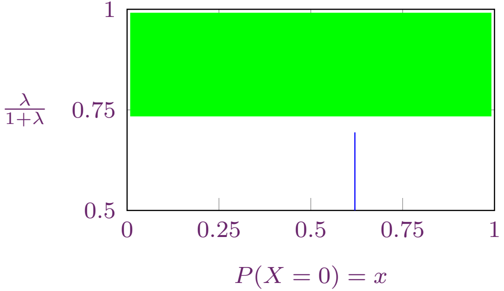


Application to binary alphabets GF(2)

Notation: $P(X=0)=x, P(Y=0|X=0)=c, P(Y=1|X=1)=d$.

Our work: When $(c-\frac{1}{2})(d-\frac{1}{2})>0$,

Example 1: $c=0.9, d=0.6$



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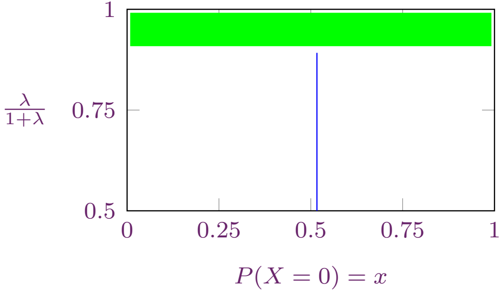
Our work: When $(c-\frac{1}{2})(d-\frac{1}{2})>0$,

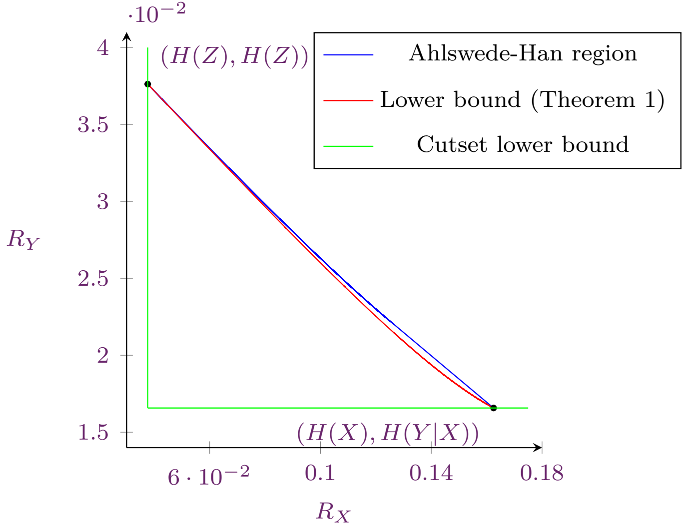
Example 2: $c=0.7, d=0.6$



Comparison of the bounds

In [Ahlswede-Han 83'], Ahlswede and Han chose the following $p(x, y)$ given by

$$p(x,y) = \begin{bmatrix} 0.003920 & 0.019920 \\ 0.976080 & 0.000080 \end{bmatrix}$$



Application to higher alphabet fields

Example 1

For $GF(3)$, one instance of $p(x,y)$ satisfying that Z is independent of Y and $\mathfrak{C}_{\mu(x)}[H(Y)-H(Z)]\big|_{p(x)} = H(Y)-H(Z)$ is given by the following distribution:

$$p(x,y) = \begin{bmatrix} 0.08 & 0.06 & 0.18 \\ 0.08 & 0.18 & 0.06 \\ 0.24 & 0.06 & 0.06 \end{bmatrix}$$

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Example 2

One instance of $p(x, y)$ satisfying $\mathfrak{C}_{\mu(x)}[H(Y) - H(Z)]\big|_{p(x)} = H(Y|X) - H(Z|X)$ is given by the following distribution:

$$p(x, y) = \begin{bmatrix} 0.02 & 0.02 & 0.48 \\ 0.02 & 0.06 & 0.16 \\ 0.06 & 0.02 & 0.16 \end{bmatrix}$$

Related open problems

Here is a conjecture verified by numerical simulations by different groups of researchers.

Conjecture [Sefidgaran-Gohari-Reza 15']

For binary random variables $X, Y, U,$ and V that satisfy the Markov chain $U - X - Y - V$, and for $Z = X \oplus Y$, we have

$$I(X, Y; U, V) + 2H(X|U, V) \geq \min\{H(X, Y), 2H(Z)\}$$

If the conjecture is true, the smallest sum-rate yielded by Ahlswede-Han region is indeed the minimum of $\{H(XY), 2H(Z)\}$, i.e. the minimum of the Slepian-Wolf region and the Körner-Marton region.

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Another open problem is whether this lower bound can be applied to Gaussian Distributed Source Coding with distortion criterion?

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