Reverse hypercontractivity region for the binary erasure channel

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Definitions

Hypercontractivity

A pair of random variables (X, Y) is said to be (λ_1, λ_2)-hypercontractive, for $\lambda_1, \lambda_2 \in (1, \infty)$, if

$$E(f(X)g(Y)) \leq ||f(X)||_{\lambda_1} ||g(Y)||_{\lambda_2}$$

holds for all non-negative functions $f(\cdot), g(\cdot)$. Here

$$||Z||_\lambda := E(|Z|^\lambda)^{\frac{1}{\lambda}}, \lambda \neq 0, \quad \text{(normalized } \lambda \text{ moment)}; \quad ||Z||_0 := e^{E(\log |Z|)}.$$

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Reverse hypercontractivity

A pair of random variables (X, Y) is said to be (λ_1, λ_2)-reverse-hypercontractive, for $\lambda_1, \lambda_2 \in (-\infty, 1)$, if

$$E(f(X)g(Y)) \ge ||f(X)||_{\lambda_1} ||g(Y)||_{\lambda_2}$$

holds for all positive functions $f(\cdot), g(\cdot)$.

Equivalent characterizations of hypercontractivity [Nair '14]

Theorem 1

Let $(X, Y) \sim \mu_{XY}$. The following assertions are equivalent:

1. For all non-negative functions $f(\cdot), g(\cdot)$,

$$E(f(X)g(Y)) \leq ||f(X)||_{\lambda_1} ||g(Y)||_{\lambda_2}$$

2. For every $\nu_{XY}(\ll \mu_{XY})$ we have (independently by [Carlen et. al. '09])

$$\frac{1}{\lambda_1} D(\nu_X || \mu_X) + \frac{1}{\lambda_2} D(\nu_Y || \mu_Y) \leq D(\nu_{XY} || \mu_{XY})$$

3. For every extension $\mu_{U|XY}$ such that $I(U; XY) > 0$ we have

$$\frac{1}{\lambda_1} I(U; X) + \frac{1}{\lambda_2} I(U; Y) \leq I(U; XY)$$

4. Let $K[f]_x$ represents the lower convex envelope of the function f evaluated at x.

$$K\left[\frac{1}{\lambda_1} H(X) + \frac{1}{\lambda_2} H(Y) - H(XY)\right]_{\mu_{XY}} = \frac{1}{\lambda_1} H(X) + \frac{1}{\lambda_2} H(Y) - H(XY)$$

Gray-Wyner region

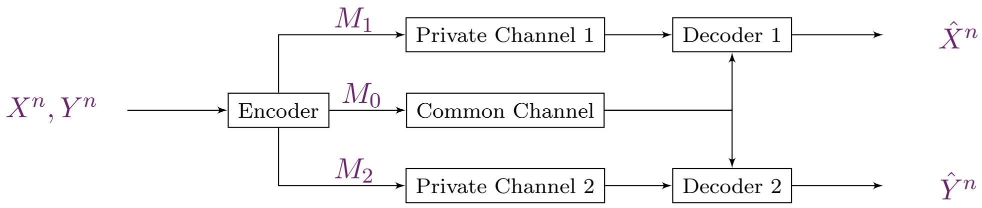

Figure 1: Gray-Wyner Network

Optimal rate region of Gray-Wyner System with 2 sources X and Y is the set of rate triples (R_0, R_1, R_2) such that

$$R_0 \geq I(XY; V),$$
$$R_1 \geq H(X|V),$$
$$R_2 \geq H(Y|V)$$

for some conditional pmf $p(v|x, y)$ with $|V| \leq |X| \cdot |Y| + 2$.

Gray-Wyner region

Computing minimum along supporting hyperplanes,

$$\begin{aligned}
&\min R_0 + \frac{1}{\lambda_1} R_1 + \frac{1}{\lambda_2} R_2 \\
=&\min I(XY;V) + \frac{1}{\lambda_1} H(X|V) + \frac{1}{\lambda_2} H(Y|V) \\
=&H(XY) + K[\frac{1}{\lambda_1} H(X) + \frac{1}{\lambda_2} H(Y) - H(XY)]_{\mu_{XY}}
\end{aligned}$$

Observations [Beigi-Gohari '15]:

- Tensorization of forward hypercontractivity $\Leftrightarrow$ Optimality of single letter expression of Gray-Wyner System
- Determining $\{\mu_{XY} : \mu_{XY} \text{ is } (\lambda_1, \lambda_2)\text{-hypercontractive}\}$ $\equiv$ Determining set of possible $\mu_{XY|V}$ for extremal distributions in the Gray-Wyner System

Equivalent characterizations of reverse hypercontractivity [Beigi-Nair '16]

Denote the reverse-hypercontractive region of (λ_1, λ_2) for a pair of random variables (X, Y) distributed according to μ_{XY} as $R^r(X, Y)$.

Theorem 2

- The pair (λ_1, λ_2) with $0 < \lambda_1 < 1, 0 < \lambda_2 < 1$ belongs to $R^r(X, Y)$ if and only if for any q_X and q_Y there exists r_{XY} with $r_X = q_X$ and $r_Y = q_Y$ such that:

$$\frac{1}{\lambda_1} D(q_X || p_X) + \frac{1}{\lambda_2} D(q_Y || p_Y) \geq D(r_{XY} || p_{XY})$$

- The pair (λ_1, λ_2) with $\lambda_1 < 0, 0 < \lambda_2 < 1$ belongs to $R^r(X, Y)$ if and only if for any q_Y there exists r_{XY} with $r_Y = q_Y$ such that:

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Applications of hypercontractivity

In Theoretical Computer Science

- Friedgut's Junta Theorem
- KKL Theorem
- Russo-Margulis formula
- sharp threshold
- small-set expansion
- stable influences
- transitive-symmetric function

In Mathematics and Physics

- Measure Concentration
- Transportation inequalities

Evaluation of (Reverse)-Hypercontractivity Parameters

Information Theory

- Related to determining *extremal auxiliaries* in multiuser information theory

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Theoretical Computer Science

- Theorem: Small set expansion hypothesis (SSEH) implies that there is no efficient approximation algorithm for the $2 \to 4$ norm. [Barak-Brandão-Harrow-Kelner-Steurer-Zhou 14']
- Corollary: If hypercontractivity parameters can be evaluated efficiently, then we can falsify SSEH.

Evaluation of (Reverse)-Hypercontractivity Parameters

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This talk: evaluation of reverse-hypercontractivity region for binary erasure channel with uniform inputs.

Known hypercontractivity parameters

Binary Symmetric Channel (BSC) with uniform input: [Bonami 70', Gross 75']

Consider a uniformly distributed binary valued X and Y obtained by passing X through a BSC with crossover probability $\frac{1-\rho}{2}$. (X, Y) is (λ_1, λ_2)- hypercontractive for $\lambda_1, \lambda_2 \in (1, \infty)$ if and only if

$$(\lambda_1 - 1)(\lambda_2 - 1) \geq \rho^2$$

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Gaussian: [Gross 75']

Let $(X,Y) \sim \mathcal{N}\left(0, \begin{bmatrix} 1 & \rho \\ \rho & 1 \end{bmatrix}\right)$, (X,Y) is (λ_1,λ_2)- hypercontractive for $\lambda_1,\lambda_2 \in (1,\infty)$ if and only if

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Known hypercontractivity parameters

Binary Erasure Channel (BEC) with uniform input: [Nair-Wang 16']

Consider a uniformly distributed binary valued X passed through a BEC with erasure probability ϵ producing the ternary output Y. When

$$\epsilon - \frac{1}{2} \leq \frac{3}{2}(\lambda_2 - 1)$$

(X, Y) is (λ_1, λ_2)- hypercontractive for $\lambda_1, \lambda_2 \in (1, \infty)$ if and only if

$$(\lambda_1 - 1)(\lambda_2 - 1) \geq 1 - \epsilon.$$

Known reverse hypercontractivity parameters

Binary Symmetric Channel (BSC) with uniform input: [Borell 82']

Consider a uniformly distributed binary valued X and Y obtained by passing X through a BSC with crossover probability $\frac{1-\rho}{2}$. (X, Y) is (λ_1, λ_2)-reverse -hypercontractive for $\lambda_1, \lambda_2 \in (-\infty, 1)$ if and only if

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Gaussian: [Borell 82']

Let $(X, Y) \sim \mathcal{N}\left(0, \begin{bmatrix} 1 & \rho \\ \rho & 1 \end{bmatrix}\right)$, (X, Y) is (λ_1, λ_2)-reverse-hypercontractive for $\lambda_1, \lambda_2 \in (-\infty, 1)$ if and only if

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Remark: In all the previous cases, the *correlation bound* is tight.

Main result

Binary Erasure Channel (BEC) with uniform input

Consider a uniformly distributed binary valued X passed through a BEC with erasure probability ϵ producing the ternary output Y. When $\lambda_2 < 0$, (X, Y) is (λ_1, λ_2)-reverse-hypercontractive if and only if

$$\lambda_1 \leq \frac{\ln 2}{\ln 2 - \frac{\lambda_2 - 1}{\lambda_2} \ln[(1-\epsilon)2^{\frac{1}{\lambda_2 - 1}} + \epsilon]}$$

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Remarks:

- The correlation bound $(\lambda_1 - 1)(\lambda_2 - 1) \geq 1 - \epsilon$ is not tight.
- However, as in all cases so far, *local analysis* suffices to compute the hypercontractivity.
- The critical behavior happens at the boundary.

(Reverse) Hypercontractive Region for BEC with uniform input

Define $\lambda_2' := \frac{\lambda_2}{\lambda_2 - 1}$, the Hölder conjugate of λ_2.
When $\epsilon = 0.2$,

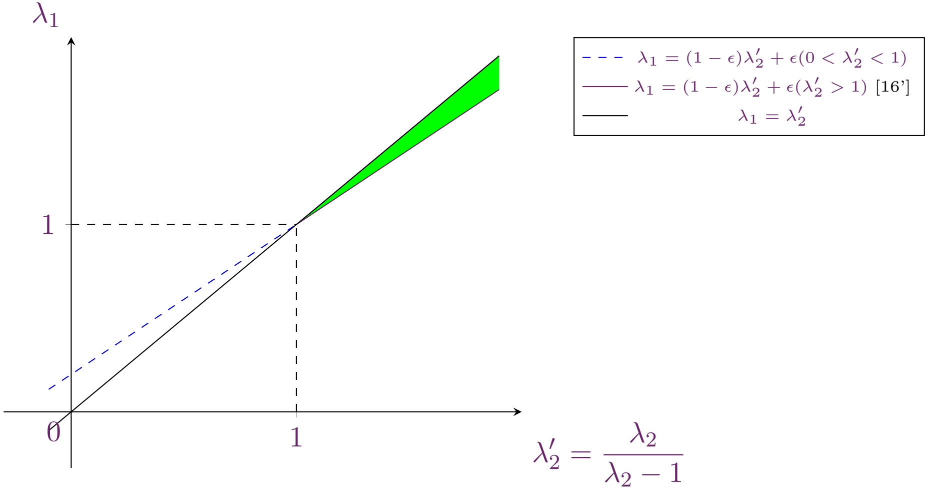

Figure 2: (Reverse) Hypercontractive Region: $\epsilon = 0.2$

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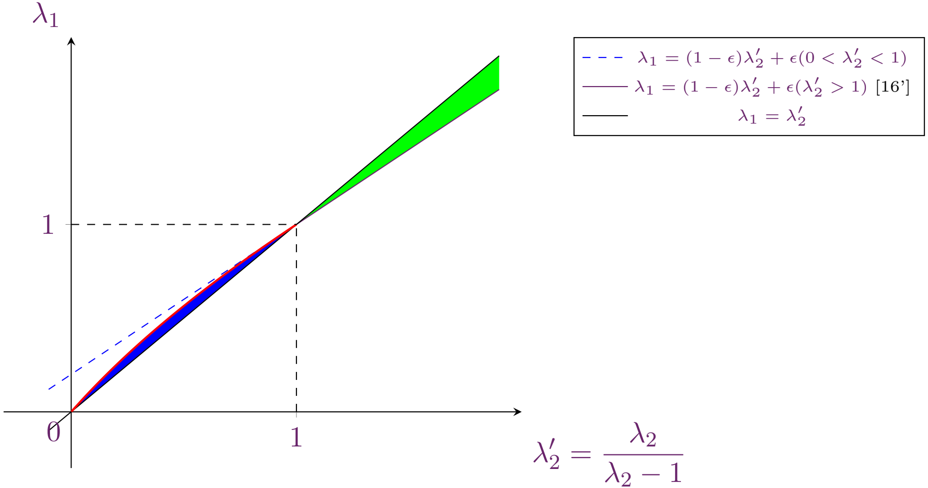

Figure 2: (Reverse) Hypercontractive Region: $\epsilon = 0.2$

Proof sketch

When $\lambda_2 < 0$, equivalent to determine (λ_1, λ_2) such that

$$\min_{q_X} \max_{r_{XY}} \frac{1}{\lambda_1} D(r_X || p_X) + \frac{1}{\lambda_2} D(q_Y || p_Y) - D(r_{XY} || p_{XY}) \geq 0$$

Define: $q_X(X=0) = x$, $r_{XY}(X=0, Y=0) = r$, $r_{XY}(X=1, Y=1) = s$.

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Under this parameterization,

$$\begin{aligned}
&\frac{1}{\lambda_1} D(r_X || p_X) + \frac{1}{\lambda_2} D(q_Y || p_Y) - D(r_{XY} || p_{XY}) \\
&= \frac{1}{\lambda_1} D\left([x, 1-x] \,||\, \left[\frac{1}{2}, \frac{1}{2}\right] \right) + \frac{1}{\lambda_2} D\left([r, 1-r-s, s] \,||\, \left[\frac{1-\epsilon}{2}, \epsilon, \frac{1-\epsilon}{2}\right] \right) \\
&\quad - D\left([r, x-r, 1-x-s, s] \,||\, \left[\frac{1-\epsilon}{2}, \frac{\epsilon}{2}, \frac{\epsilon}{2}, \frac{1-\epsilon}{2}\right] \right) \\
&=: f(x, r, s)
\end{aligned}$$

Wish to determine (λ_1, λ_2) (with $\lambda_2 < 0$) such that

$$\min_{x \in [0,1]} \max_{r,s : r \in [0,x], s \in [0,1-x]} f(x, r, s) \geq 0.$$

Proof sketch: continued

Define, for $x \in [0,1]$

$$g(x) := \max_{r,s: r \in [0,x], s \in [0,1-x]} f(x,r,s).$$

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Easy direction: From above, we require $g(0) \geq 0$. This implies that

$$\lambda_1 \leq \frac{\ln 2}{\ln 2 - \frac{\lambda_2 - 1}{\lambda_2} \ln[(1-\epsilon)2^{\frac{1}{\lambda_2 - 1}} + \epsilon]}$$

Remark: This boundary condition is stronger than correlation bound for $0 < \epsilon < 1$.

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For the non-trivial direction, we show that

- $g(x)$ is symmetric along $x = \frac{1}{2}$. (Easy - by symmetry of the (X,Y)-distribution)
- $g(x)$ has only one stationary point, i.e. $g'(x) = 0$, between $(0, \frac{1}{2})$.
- $g(x)$ is convex at $x = \frac{1}{2}$ and $g'\left(\frac{1}{2}\right) = 0, g\left(\frac{1}{2}\right) = 0$. (Easy)

$g(x)$ has only one stationary point between $(0, \frac{1}{2})$

- Recall $g(x) := \max_{r,s:r\in[0,x],s\in[0,1-x]} f(x,r,s)$.
  Hence any stationary point of $g(x)$ will be a stationary point of $f(x,r,s)$.

- Let $y = \frac{2(x-r)}{1-r-s}$; it is easy to show that

  the stationary points of $f(x,r,s)$ are in 1-1 correspondence with the roots of

$$\frac{1-\epsilon}{\epsilon} y^{\lambda_2'-\lambda_1} + y^{1-\lambda_1} = \frac{1-\epsilon}{\epsilon}(2-y)^{\lambda_2'-\lambda_1} + (2-y)^{1-\lambda_1}.$$

  Hence suffices to show that there is exactly *one root* of above equation for $y \in (0,1)$.

One root of the equation: continued

Define $h(y) = \frac{1-\epsilon}{\epsilon} y^{\lambda_2'-\lambda_1} + y^{1-\lambda_1} - \frac{1-\epsilon}{\epsilon}(2-y)^{\lambda_2'-\lambda_1} - (2-y)^{1-\lambda_1}$.

$\lim_{y \downarrow 0} h(y) = +\infty$ and $\lim_{y \downarrow 0} h'(y) = -\infty$.

On the other hand $h(1) = 0$ and $h'(1) = 2\frac{(1-\epsilon)\lambda_2' + \epsilon - \lambda_1}{\epsilon} > 0$ ($\because (\lambda_1 - 1)(\lambda_2 - 1) > 1 - \epsilon$). Thus $h(y) = 0$ has at least one root for $y \in (0, 1)$.

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To show that $h(y) = 0$ has exactly one root, suffices to show that $h'(y) = 0$ *has at most one root in* $(0, 1)$.

By Taylor expansion,

$$h'(1-t) = 2\sum_{k=0}^{\infty} \left[\frac{(1-\epsilon)(\lambda_2' - \lambda_1)}{\epsilon} \binom{\lambda_2' - \lambda_1 - 1}{2k} + (1-\lambda_1)\binom{-\lambda_1}{2k} \right] t^{2k}$$

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Key observation: The coefficients of t^{2k} are: nonnegative for $k \le k_0$, and negative for $k > k_0$ for some k_0 (one sign-change). □

Recap

Our analysis showed that:

- Only one interior local minimum (of $g(x)$) when correlation bound is strictly satisfied.
- Hence suffices to study the behavior at boundary. (For BEC, the critical case)

We also recover Borell's result (BSC) in the paper using similar ideas (easier analysis).

Recap

Our analysis showed that:

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- Hence suffices to study the behavior at boundary. (For BEC, the critical case)

We also recover Borell's result (BSC) in the paper using similar ideas (easier analysis).

Similar *local* arguments used earlier for determining certain extremal auxiliaries in network information theory.
For instance, the proof of the inequality

$$I(U;Y) + I(V;Z) - I(U;V) \leq \max\{I(X;Y), I(X;Z)\}$$

when $|X| = 2$ and $(U,V) - X - (Y,Z)$ is Markov.

Similar *local* analysis (to identify the extremal distributions) also works in forward hypercontractivity proofs of BSC and BEC.

Related Open Questions

Does such *local* analysis work in general for determining hypercontactivity parameters?

- In other words, does the functional

$$\frac{1}{\lambda_1} H(X) + \frac{1}{\lambda_2} H(Y) - H(XY)$$

  have *nice* geometric properties that allow such local arguments to work
- If so, can we devise an algorithm to efficiently approximate the hypercontractivity parameters?

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- If so, can we devise an algorithm to efficiently approximate the hypercontractivity parameters?

Here is an open problem from Network Information Theory where simulations suggest that such *local arguments* should suffice.

Conjecture [Sefidgaran-Gohari-Reza 15']

For binary random variables X, Y, U, and V that satisfy the Markov chain $U - X - Y - V$, and for $Z = X \oplus Y$, we have

$$I(X,Y;U,V) + 2H(X|U,V) \geq \min\{H(X,Y), 2H(Z)\}$$

References

- S. Beigi and C. Nair, *Equivalent characterization of reverse brascamp-lieb-type inequalities using information measures*, 2016 IEEE International Symposium on Information Theory (ISIT), July 2016, pp. 1038–1042.
- Salman Beigi and Amin Gohari, *On the duality of additivity and tensorization*, CoRR **abs/1502.00827** (2015).
- Boaz Barak et al., *Hypercontractivity, sum-of-squares proofs, and their applications*, Proceedings of the forty-fourth annual ACM symposium on Theory of computing, ACM, 2012. pp. 307-326.
- Aline Bonami, *Étude des coefficients de fourier des fonctions de $l^p(g)$*, Annales de l'institut Fourier **20** (1970), no. 2, 335–402 (fre).
- Christer Borell, *Positivity improving operators and hypercontractivity.*, Mathematische Zeitschrift **180** (1982), 225–234.
- Leonard Gross, *Logarithmic Sobolev inequalities*, Amer. J. Math. **97** (1975), no. 4, 1061–1083. MR 0420249 (54 #8263)
- C. Nair and Y. N. Wang, *Evaluating hypercontractivity parameters using information measures*, 2016 IEEE International Symposium on Information Theory (ISIT), July 2016, pp. 570–574.
- M. Sefidgaran, A. Gohari, and M. R. Aref, *On korner-marton's sum modulo two problem*, 2015 Iran Workshop on Communication and Information Theory (IWCIT), May 2015, pp. 1–6.